Last Time: Curl and Divergence

$\text{curl}(\vec{v}) = \nabla \times \vec{v}$ with $\vec{v} = \langle P, Q, R \rangle$; $\text{div}(\vec{v}) = \nabla \cdot \vec{v}$

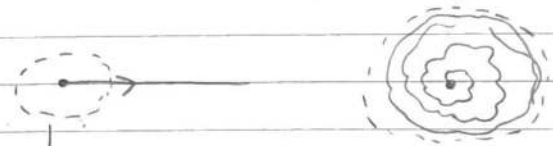
Prop: ① $\text{curl}(\nabla f) = \vec{0}$ ② $\text{div}(\text{curl}(\vec{v})) = 0$

Interpretations of Curl and Divergence:

① Curl measures "how swirly is the v.f."?

↳ $\text{curl}(\vec{v})$ is always "swirly".

② Divergence measures "does the v.f. tend to push points away from a little open region"?

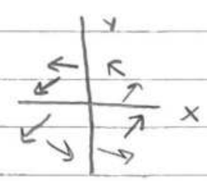


Ex: Consider v.f. $\langle P(x,y), Q(x,y), 0 \rangle = \vec{v}$

$$\text{curl}(\vec{v}) = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{bmatrix}$$

$$= \langle -Q_z, +P_z, Q_x - P_y \rangle$$

$$= \langle 0, 0, Q_x - P_y \rangle = \langle 0, 0, \underbrace{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}}_{} \rangle \; *$$



view from above

Recasting Green's Theorem w/ vector Feilds:

Let $\vec{v} = \langle P(x,y), Q(x,y), 0 \rangle$ have cts. partial derivatives on some open region $R \subseteq \mathbb{R}^2$ and containing a closed region D w/ peicewise smooth, simple, closed boundary curve.

and ① $\iint_D \text{curl}(\vec{v}) \cdot \vec{k} \, dA = \int_{\partial D} \vec{v} \cdot d\vec{r}$

② $\int_{\partial D} \vec{v} \cdot (y'(t)\vec{i} - x'(t)\vec{j}) \frac{1}{|\vec{r}'(t)|} ds = \iint_D \text{div}(\vec{v}) \, dA$

Why:

① $\operatorname{curl}(\vec{v}) = \langle 0, 0, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \rangle$, so $\operatorname{curl}(\vec{v}) \cdot \vec{k} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$

$$\therefore \iint_D \operatorname{curl}(\vec{v}) \cdot \vec{k}\, dA = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$\underset{\text{Green's Theorem}}{=} \int_{\partial D} P\,dx + Q\,dy$$

$$= \int_{t=a}^{b} \left(P(x,y)\, x'(t) + Q(x,y)\, y'(t) \right) dt$$

$$= \int_{t=a}^{b} \langle P, Q, 0 \rangle \cdot \langle x', y', z' \rangle\, dt$$

$$= \int_{\partial D} \vec{v} \cdot d\vec{r}$$

② $\iint_D \operatorname{div}(\vec{v})\, dA = \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dA \qquad \omega = \langle \underset{A}{-Q}, \underset{B}{P}, 0 \rangle$

$$= \iint_D \left(\frac{\partial P}{\partial x} - \frac{\partial (-Q)}{\partial y} \right) dA \longleftrightarrow \iint_D \left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y} \right) dA$$

$$= \int_{\partial D} -Q\,dx + P\,dy \longleftrightarrow \int_{\partial D} A\,dx + B\,dy$$

$$= \int_{t=a}^{b} (-Qx' + Py')\, dt$$

$$= \int_{t=a}^{b} (Py' - Qx')\, dt$$

$$= \int_{t=a}^{b} \langle P, Q \rangle \cdot \langle y', -x' \rangle\, dt$$

$$= \int_{\partial D} \vec{v} \cdot (y'(t)\vec{i} - x'(t)\vec{j}) \frac{1}{|\vec{r}'(t)|}\, ds$$

<u>NB</u>: These two ways of rewriting Green's Theorem with <u>① curl</u> and <u>② divergence</u> are jumping points for generalizing Green's theorem

①Generalizing using Curl: Stoke's Theorem

②Generalizing using divergence: Divergence Theorem

Below this line is <u>not</u> on exam 3, but will be on final:

<u>Section 16.6: Parametric surfaces</u>:

<u>Idea</u>: Generalize space curves to have dimension 2...

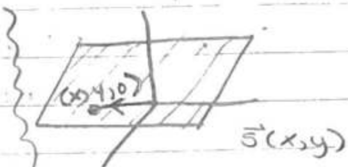
<u>Def^n</u>: A parametric surface in 3-space is given by a vector function:

$$\vec{S}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle \text{ on}$$

some domain $D \subseteq \mathbb{R}^2$.

Ex: The Euclidean Plane sits in $\mathbb{R}^3$ as a parametric surface:

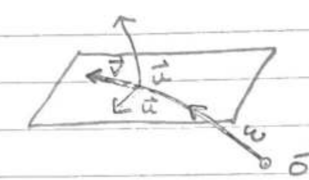
$\vec{s}(x,y) = \langle x, y, 0 \rangle$ on $D = \mathbb{R}^2$



Ex: Every plane Π in $\mathbb{R}^3$ can be parameterized in a similar way:

$\vec{s}(a,b) = a\vec{u} + b\vec{v} + \vec{w}$ for suitable $\vec{u}, \vec{v}, \vec{w}$ on $D = \mathbb{R}^2$

I.e. $\vec{s}(a,b) = \langle u_1 a + v_1 b + w_1,\ u_2 a + v_2 b + w_2,\ u_3 a + v_3 b + w_3 \rangle$



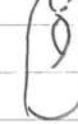
Ex: The sphere of radius $r > 0$ is parameterized by:

$$\vec{s}(\theta, \varphi) = \langle r\sin(\varphi)\cos(\theta), r\sin(\varphi)\sin(\theta), r\cos(\varphi) \rangle \text{ on } D = [0, 2\pi] \times [0, \pi]$$

Ex: The torus has parameterization

$$\vec{s}(\theta, \varphi) = \langle (2 + \sin(\theta))\cos(\varphi), (2 + \sin(\theta))\sin(\varphi), \cos(\theta) \rangle$$

on $D = [0, 4\pi] \times [0, 2\pi]$



Ex: Parameterize the paraboloid $z = x^2 + 2y^2$

* NB: There is no one parameterization of a surface*

Sol: ① $\vec{s}(x,y) = \langle x, y, x^2 + 2y^2 \rangle$ on $D = \mathbb{R}^2$

Sol: ② $\vec{s}(r,\theta) = \langle r\cos\theta, r\sin\theta, (r\cos\theta)^2 + 2(r\sin\theta)^2 \rangle$

$= \langle r\cos\theta, r\sin\theta, r^2(1 + \sin^2\theta) \rangle$

on $D = [0, \infty) \times [0, 2\pi]$

Sol: ③ $\vec{s}(r,\theta) = \langle \sqrt{2}r\cos\theta, r\sin\theta, r^2 \rangle$ on

$D = [0, \infty) \times [0, 2\pi]$ ▨.

Ex: A surface of revolution (about x-axis) be obtained for a function $f(x)$ via

$\vec{s}(x, \theta) = \langle x, f(x)\cos\theta, f(x)\sin\theta \rangle$ on $D = \mathrm{dom}(f) \times [0, 2\pi]$. ▨.

$f(x) = x + 1$